Physica B ▮ (▮▮▮▮) ▮▮▮–▮▮▮

Contents lists available at ScienceDirect

Physica B

journal homepage: www.elsevier.com/locate/physb

On the definition of the Poynting vector: A non-local derivation

Rubén G. Barrera [a,*], W. Luis Mochán [b], Augusto García-Valenzuela [c], Edahí Gutiérrez-Reyes [a]

[a] Instituto de Física, Universidad Nacional Autónoma de México, AP 20-364, 01000 México DF, Mexico
[b] Instituto de Ciencias Físicas, Universidad Nacional Autónoma de México, AP 48-3, 62250 Cuernavaca, Mor., Mexico
[c] Centro de Ciencias Aplicadas y Desarrollo Tecnológico, Universidad Nacional Autónoma de México, AP 70-168, 04510 México DF, Mexico

ABSTRACT

We review a formalism developed previously to deal with the non-local (spatially dispersive) effective electromagnetic response of colloidal systems. We use this formalism to find, in the long wavelength limit, a general expression for the local energy conservation theorem in terms of parameters associated to the non-local response of the system. We find an explicit expression for the Poynting vector in terms of these non-local parameters and show that there are several ways to write it in terms of the electric permittivity and the magnetic permeability. The significance of this problem in the treatment of refraction is also discussed.

1. Introduction

The flux of electromagnetic energy through space is defined in terms of Poynting's vector (PV). An explicit expression for PV in terms of the electric and magnetic fields is obtained by deriving, from Maxwell's equations, a relation that can be identified as a local energy conservation law. Although it has long been recognized that a local conservation law does not provide a unique identification of the electromagnetic energy flux, it has been accepted that the energy flux in free space is given, in SI units, by $\vec{E}\times\vec{B}/\mu_0$, where $\vec{E}$ is the electric field, $\vec{B}$ is the magnetic field and μ_0 is the so called permeability of vacuum. However there are further possible definitions of PV within material bodies. It was proposed, long ago [1], that in order to avoid energy accumulation at the boundary between two media one should define PV as $\vec{E}\times\vec{H}$, recalling that the tangential components of $\vec{H}$ are continuous across the interface. This has become the standard choice. However, in the high-frequency range the choice of the correct expression for PV had not been given much attention because in this frequency range the difference between $\vec{B}/\mu_0$ and $\vec{H}$, in common materials, is negligible. However, due to the recent activity in negative refraction, metamaterials and colloidal systems the distinction between $\vec{B}/\mu_0$ and $\vec{H}$ becomes fundamental.

Here we start by reviewing the non-local formalism and the main results obtained in our recently-developed effective-medium treatment of colloidal systems. Then we use this non-local formalism to derive Poynting's theorem for a quasi-monochromatic pulse and consider frequencies at which the system has a negligible absorption. Then we take the long wavelength limit. The main advantage of this approach is that in the non-local formalism the magnetic effects are incorporated into the *generalized* non-local dielectric tensor $\overleftrightarrow{\varepsilon}_g(k,\omega)$ of the system. Within this approach there is no formal difference between $\vec{B}$ and $\vec{H}$, that is $\vec{B}=\mu_0\vec{H}$. Poynting's vector can then be expressed in terms of this *generalized* non-local dielectric response. In the long wavelength limit we keep the quadratic coefficients of $\overleftrightarrow{\varepsilon}_g(k,\omega)$ when expanded in powers of the wave vector k. Due to the intrinsic non-local character of the magnetic response, the expressions obtained here in the long wavelength limit ($k\to 0$) have a very general validity and are not restricted to colloidal systems with small inclusions. Finally we relate the expansion coefficients of the non-local response to the usual local permittivity $\varepsilon(\omega)$ and local permeability $\mu(\omega)$ and show that there is not a unique way to express PV in terms of them. The lack of uniqueness arises from the freedom in the choice of $\varepsilon(\omega)$ and $\mu(\omega)$.

Colloids are very complex systems usually defined, in their simplest form, as a dispersed phase embedded within a homogeneous one. Typical examples of these systems are milk, clouds, blood, paints, fog, and ink. The dispersed phase consists of small colloidal inclusions located randomly in space. When light enters into these systems it gets scattered and absorbed by the inclusions. The description and understanding of how energy is transported within these systems, and how light gets refracted and reflected in the presence of a "flat" interface is a problem that has attracted the interest of many researchers for quite a long time [2].

When the size of the inclusions is not too small compared to the wavelength of the incident radiation, scattering gives rise to an electromagnetic field that propagates in all directions. This diffuse field is responsible for the turbid appearance of some colloids like milk, blood and clouds. Nevertheless, due to

* Corresponding author. Tel.: +52 55 5622 5093; fax: +52 55 5616 1535.
E-mail address: rbarrera@fisica.unam.mx (R.G. Barrera).

0921-4526/$ - see front matter © 2010 Published by Elsevier B.V.
doi:10.1016/j.physb.2010.01.004

constructive interference, there is also a field that propagates in the direction of the incident radiation. This is usually called the coherent beam or the average field. Due to the presence of both fields, it is not clear *a priori* if a concept like an effective index of refraction would make any sense.

Nevertheless, if we focus our attention on the coherent beam, we realize that this beam propagates in a single direction and gets refracted and reflected at interfaces, suggesting that this behavior could indeed be described in terms of an effective medium with appropriate optical properties like an effective index of refraction. Recently we have shown that this is indeed possible, but the effective medium turns out to be spatially dispersive [3], that is, its effective optical properties depend not only on the frequency of the external exciting source but also on its wave vector. This corresponds to a non-local electromagnetic response of the system in real space. Furthermore, we also found [3] that there might be a sizable effective magnetic permeability at optical frequencies coming from closed currents induced within the inclusions. Although it is now clear that one may extend effective-medium theories to situations in which turbidity is strong, to keep things simple while gaining a full understanding of PV in magnetic media, we will focus our attention to the case where turbidity is negligible but non-locality is still important yielding induced magnetic properties at optical frequencies. In this respect, colloids are analogous to magnetic metamaterials.

2. Formalism

Here we introduce the effective-medium formalism necessary to deal with spatially dispersive (non-local) systems and show that refraction experiments are probably the most convenient way to determine their bulk effective properties. The properties of an effective medium characterizing an inhomogeneous system are defined in terms of appropriate averages of the fields and the currents induced by an external source. By "average" we mean a procedure that irons out the rapid spatial variations of the electromagnetic field due to the presence of the inhomogeneities down to a certain length scale, while the word "appropriate" refers to the adequate correspondence between the averaging procedure and the quantities that are actually measured.

As an example of a non-local electromagnetic response, let us take the relation between the average of the total induced current $\vec{J}_{ind}$ and the average electric field $\vec{E}$ (Ohm's law). This relation is referred as being non-local whenever the current density induced at point $\vec{r}$ depends not only on the value of the average electric field at $\vec{r}$, but also on its values around $\vec{r}$ (non-local Ohm's law), that is,

$$\vec{J}_{ind}(\vec{r};t)=\int dt' \int d^3r' \overleftrightarrow{\sigma}_g(\vec{r}-\vec{r}';t-t')\cdot\vec{E}(\vec{r}';t'), \tag{1}$$

where $\overleftrightarrow{\sigma}_g$ is the *generalized* non-local effective conductivity tensor. Similarly, its non-local dependence on $t-t'$ accounts for the non-instantaneous character of the response that yields the usual dependence on frequency usually referred simply as "dispersion", or more precisely, as "time dispersion". The word *generalized* is attached to $\overleftrightarrow{\sigma}_g$ because $\vec{J}_{ind}$ includes all contributions to the induced currents, even those that are usually interpreted in terms of the magnetization.

Even when the current induced at a point $\vec{r}$ within an inclusion responds locally to the total electric field at the same point $\vec{r}$, it might respond non-locally to the external field, as the latter has no information about the size and shape of the inclusion, information that is incorporated by the *generalized* non-local conductivity tensor. If the inclusions were small enough spheres or ellipsoids, then both the total and the external electric fields would be spatially constant within the inclusion, and therefore, proportional to each other. The proportionality constant would have the information about the size and shape of the inclusion. In this case, the induced current would be algebraically proportional to the external field and described with a local response function. Nevertheless, when the inclusions are so large that the spatial variations of the external and the total electric fields within the inclusion are non-negligible, an algebraic proportionality would no longer hold. Instead, we would require a non-local response whose kernel is the *generalized* non-local conductivity tensor.

We have shown [3] that in the case of dilute colloids regarded as a collection of randomly located spherical inclusions, the averaging procedure preserves some of the non-local character of the response of the individual inclusions. Furthermore, for systems that are homogeneous "on the average", the Fourier transform of the effective conductivity tensor depends independently on both the frequency ω and the wave vector k. By homogeneous "on the average" we mean that the probability density for finding an inclusion centered at $\vec{r}$ is translationally invariant. If the system is also isotropic "on the average" the *generalized* effective conductivity tensor can be written as

$$\overleftrightarrow{\sigma}_g(\vec{k},\omega)=\sigma_g^L(k,\omega)\overleftrightarrow{P}^L+\sigma_g^T(k,\omega)\overleftrightarrow{P}^T \tag{2}$$

in terms of only two scalar functions $\sigma_g^L(k,\omega)$ and $\sigma_g^T(k,\omega)$, the *generalized* longitudinal and transverse conductivities. The tensors $\vec{P}^L\equiv\vec{k}\vec{k}/k^2$ and $\vec{P}^T=\overleftrightarrow{1}-\vec{k}\vec{k}/k^2=-\vec{k}\times\vec{k}\times/k^2$ are the longitudinal and transverse projection operators, respectively. The corresponding *generalized* non-local effective permittivity is

$$\overleftrightarrow{\varepsilon}_g(\vec{k},\omega)=\overleftrightarrow{1}\,\varepsilon_0+\frac{i}{\omega}\overleftrightarrow{\sigma}_g(\vec{k},\omega)=\varepsilon_g^L(k,\omega)\overleftrightarrow{P}^L+\varepsilon_g^T(k,\omega)\vec{P}^T \tag{3}$$

We will call this form of expressing the electromagnetic response as the LT scheme. As mentioned above, $\vec{J}_{ind}$ includes all contributions to the induced currents, even those that are usually interpreted in terms of the magnetization. Therefore, the magnetic response may be expressed in terms of ε_g^L and ε_g^T. It can also be shown that in the long wavelength limit ($k\to0$) ε_g^L and ε_g^T have a quadratic dependence on k, that is

$$\varepsilon_g^L(k,\omega)=\varepsilon_g^{[0]}(\omega)+\varepsilon_g^{L[2]}(\omega)\frac{k^2}{k_0^2}+\ldots \quad \text{and}$$
$$\varepsilon_g^T(k,\omega)=\varepsilon_g^{[0]}(\omega)+\varepsilon_g^{T[2]}(\omega)\frac{k^2}{k_0^2}+\ldots, \tag{4}$$

and both tend to the same value $\varepsilon_g^{[0]}(\omega)$ in the $k\to0$ limit. Here $k_0^2\equiv\omega^2\varepsilon_0\mu_0$.

One of the main characteristic features of non-local systems is the possible existence of longitudinal free-propagating modes, with a dispersion relation

$$\varepsilon_g^L(k,\omega)=0, \tag{5}$$

as obtained from Gauss' law, besides the usual transverse free-propagating modes, whose dispersion relation can be written as,

$$k^2=k_0^2\tilde{\varepsilon}_g^T(k,\omega), \tag{6}$$

where the tilde on top of a symbol indicates dimensionless quantity, i.e. $\tilde{\varepsilon}\equiv\varepsilon/\varepsilon_0$. One can now introduce the concept of an effective index of refraction by solving first Eq. (6) for k as a function of ω. Denoting this solution by $k^T(\omega)$, the effective index of refraction index can be defined as

$$n_{eff}(\omega)=\sqrt{\tilde{\varepsilon}_g^T(k^T(\omega),\omega)}, \tag{7}$$

which is a function of frequency only. Note that: (i) the solution for k in Eqs. (5) and (6) might be a complex quantity, where

Please cite this article as: R.G. Barrera, et al., Physica B (2010), doi:10.1016/j.physb.2010.01.004

$k=\sqrt{\vec{k}\cdot\vec{k}}=k'+ik''$, (ii) there might be several different solutions of Eqs. (5) and (6), thus in the case of transverse modes one could have several effective indices of refraction, one for each mode, (iii) since ε_g^T is in general complex, $n_{eff}(\omega)$ is also, in general, a complex quantity, and (iv) $n_{eff}(\omega)$ may not be substituted for the usual index of refraction of local optics in reflection and refraction problems due to its non-local origin; its applicability should be carefully examined first.

The experimental determination of $n_{eff}(\omega)$ would be extremely useful for the characterization of the colloidal inclusions: their size and shape as well as their filling fraction. Nevertheless, most of the experimental techniques used today in local optics for the determination of the index of refraction, are based on the inversion of reflection experiments. In our non-local case the use of reflection complicates the inversion problem because non-locality translational symmetry is lost and there is necessarily a surface region that responds differently than the bulk. Thus the reflection amplitudes depend not only on $n_{eff}(\omega)$, but also on the structure of the surface region.

A convenient alternative would be to use the geometry of refraction instead of the reflection coefficients. It is less demanding because the angle subtended by the energy flux beyond the surface region would not be affected by its structure. Although the experimental set up will require some precautions, we believe that it will be possible to develop a coherent-refraction spectroscopy that could yield reliable values of $n_{eff}(\omega)$ in colloids.

The main ingredient in the refraction phenomenon is the direction of the energy flux (PV). There has been a revival of the refraction problem in local optics due to the advent of negative refraction and the development of metamaterials [4]. Nevertheless, there has been discussion about the interpretation of some negative-refraction experiments [5]. Moreover, the mere existence of negative refraction has been recently challenged [6] by claiming that the expression commonly used for the calculation of the energy flux (PV) is not correct. Conversely, there are also reports of direct experimental confirmation of negative refraction at optical frequencies [7]. Thus a consensus on this matter has not yet been achieved.

Here we deal with the issue related to the correct definition of PV from our non-local perspective. As induced magnetism is an intrinsically non-local effect, our non-local formalism should yield the correct expression for the energy flux. We also think that our non-local perspective should clarify some aspects about the physics behind this problem.

3. The energy theorem

We consider a quasi-monochromatic electromagnetic plane wave-packet with an electric field given by

$$\vec{E}=\mathrm{Re}\,\vec{E}_0(\vec{r},t)\exp[i\vec{k}\cdot\vec{r}-\omega t] \tag{8}$$

where $\vec{k}$ and ω are real, fixed, independent quantities, $\vec{E}_0(\vec{r},t)$ is in general a complex slowly varying function of $\vec{r}$ and t. By this we mean

$$|\nabla E_{0\alpha}|\ll kE_0 \quad \text{and} \quad \left|\frac{\partial E_{0\alpha}}{\partial t}\right|\ll \omega E_0 \tag{9}$$

where the subscript α denotes cartesian component. By using Maxwell's equations, we derive immediately Poynting's theorem

$$\nabla\cdot\frac{1}{2}\,\mathrm{Re}\left(\vec{E}\times\frac{\vec{B}^*}{\mu_0}\right)+\frac{\partial}{\partial t}\left(\frac{\varepsilon_0}{2}|E|^2+\frac{1}{2\mu_0}|B|^2\right)=-W_{ext}-W_{ind}, \tag{10}$$

where * denotes complex conjugate and we have taken an average over time. Eq. (10) has the explicit form required by a local conservation law, consisting of the relation between the divergence of a flux and the time variation of the conserved quantity. Here Poynting's vector $\vec{S}=\mathrm{Re}(\vec{E}\times\vec{B}^*)/\mu_0$ is the energy flux and $u=\varepsilon_0|E|^2/2+|B|^2/2\mu_0$ the energy density, $W_{ext}\equiv\mathrm{Re}(\vec{J}_{ext}\cdot\vec{E}^*_{ext})/2$ denotes the work done on the external currents that keep a fixed wave vector $\vec{k}$, frequency ω and amplitude $\vec{E}_0(\vec{r},t)$ of the quasi-monochromatic pulse, while $W_{ind}=\mathrm{Re}(\vec{J}_{ind}\cdot\vec{E}^*)/2$ denotes the work done on the induced currents, where $\vec{J}_{ind}$ denotes all possible currents induced through all possible mechanisms. W_{ind} is sometimes identified with dissipated heat, but we will show below that this is not always the case.

We do this by first computing $\vec{J}_{ind}$ using Eq. (1), with the electric field given by Eq. (8) and using the relation between $\sigma_{g,\alpha\beta}$ and $\varepsilon_{g,\alpha\beta}$ given in Eq. (3). One gets, following Refs. [8;1, Chap. XII],

$$\begin{aligned}J_{ind,a}(\vec{r},t)=\exp[i(\vec{k}\cdot\vec{r}-\omega t)]\Bigg\{&-i\omega(\varepsilon_{g,\alpha\beta}(\vec{k},\omega)-\varepsilon_0\delta_{\alpha\beta})E_{0\beta}(r,t)\\&-\omega\frac{\partial\varepsilon_{g,\alpha\beta}(\vec{k},\omega)}{\partial k_\gamma}\frac{\partial E_{0\beta}(\vec{r},t)}{\partial x_\gamma}\\&+\left[(\varepsilon_{g,\alpha\beta}-\varepsilon_0\delta_{\alpha\beta})+\omega\frac{\partial\varepsilon_{g,\alpha\beta}(\vec{k},\omega)}{\partial\omega}\right]\frac{\partial E_{0\beta}(\vec{r},t)}{\partial t}\Bigg\}\end{aligned} \tag{11}$$

where a first order Taylor expansion of the slowly varying field amplitudes has been made. Now we calculate W_{ind} in the long wavelength limit by using the expression for $\varepsilon_{g,\alpha\beta}$ given in Eq. (3), the second order expansions of ε_g^L and ε_g^T given in Eq. (4), and for simplicity in the interpretation we also assume negligible absorption, that is, $\mathrm{Im}\varepsilon_g^L\ll\mathrm{Re}\varepsilon_g^L$ and $\mathrm{Im}\varepsilon_g^T\ll\mathrm{Re}\varepsilon_g^T$. One obtains

$$\begin{aligned}W_{ind}=\frac{\omega}{2}[\mathrm{Im}\varepsilon_g^L(k,\omega)|E_0^L|^2+\mathrm{Im}\varepsilon_g^T(k,\omega)|E_0^T|^2]-\nabla\cdot\frac{1}{2}\mathrm{Re}\left[\frac{1}{\varepsilon_0\mu_0}\right](\vec{E}_0\times\vec{B}_0^*)\varepsilon_g^{T[2]*}(\omega)\\+\frac{\varepsilon_g^{L[2]}(\omega)}{\omega\varepsilon_0\mu_0}(\vec{k}|E_0^L|^2+k\vec{E}_0^{L*})+\frac{1}{2}\frac{\partial}{\partial t}\mathrm{Re}\left[\frac{\partial\omega\varepsilon_g^{[0]}(\omega)}{\partial\omega}|E_0|^2\right.\\\left.+\frac{k^2}{\varepsilon_0\mu_0}\left(\frac{\partial}{\partial\omega}\frac{\varepsilon_g^{L[2]}(\omega)}{\omega}|E_0^L|^2+\frac{\partial}{\partial\omega}\frac{\varepsilon_g^{T[2]}(\omega)}{\omega}|E_0^T|^2\right)-\varepsilon_0|E_0|^2\right]\end{aligned} \tag{12}$$

where $\vec{E}_0^L=\vec{P}^L\cdot\vec{E}_0(\vec{r},t)$ and $\vec{E}_0^T=\vec{P}^T\cdot\vec{E}_0(\vec{r},t)$ are the longitudinal and transverse components of the electric field, respectively. Note that in these definitions we are using the restrictions imposed in Eq. (9) thus $\nabla\cdot\vec{E}\simeq\vec{k}\cdot\vec{E}_0$. For economy in the notation we have kept ε_g^L and ε_g^T unexpanded in the first term on the r. h. s. of Eq. (12) while in the second term we have introduced the magnetic field through $\vec{k}\times\vec{E}_0=\omega\vec{B}_0$, using Faraday's law and Eq. (9). One can see that W_{ind} cannot be interpreted simply as dissipated heat because some of its terms correspond to the divergence of an additional contribution to the energy flux, i.e. in the non-local case some of the energy absorbed at a given position by the induced current may be transported to another position instead of being dissipated.

We substitute Eq. (12) into Poynting's theorem, Eq. (10), for the case when the electric field is purely longitudinal and we keep the lowest order terms in k, to obtain

$$\begin{aligned}\nabla\cdot\frac{1}{2}\mathrm{Re}\left(-\frac{\varepsilon_g^{L[2]}(\omega)}{\omega\varepsilon_0\mu_0}\vec{k}|E_0|^2\right)+\frac{1}{2}\frac{\partial}{\partial t}\mathrm{Re}\frac{\partial\omega\varepsilon_g^{[0]}(\omega)}{\partial\omega}|E_0|^2\\=-W_{ext}-\frac{\omega}{2}\mathrm{Im}\varepsilon_g^{[0]}(\omega)|E_0|^2.\end{aligned} \tag{13}$$

We see that the energy flux

$$\vec{S}_{LONG}=-\frac{1}{2}\mathrm{Re}\left(-\frac{\varepsilon_g^{L[2]}(\omega)}{\omega\varepsilon_0\mu_0}\vec{k}|E_0|^2\right) \tag{14}$$

Please cite this article as: R.G. Barrera, et al., Physica B (2010), doi:10.1016/j.physb.2010.01.004

for a longitudinal field has a non-local character, being proportional to the second order coefficient $\varepsilon_g^{L[2]}(\omega)$, of the k-expansion of ε_g^L. This is in agreement with the well known fact that the propagation of longitudinal modes is only possible in the presence of spatial dispersion.

Similarly, in the case of a pure transverse field, that is, $\nabla\cdot\vec{E}\simeq\vec{k}\cdot\vec{E}_0=0$, one can write

$$\nabla\cdot\frac{1}{2}\mathrm{Re}\left[\frac{1}{\mu_0}(\vec{E}_0\times\vec{B}_0^*)(1-\varepsilon_g^{T[2]*}(\omega)/\varepsilon_0)\right]+\frac{1}{2}\frac{\partial}{\partial t}\mathrm{Re}\left[\frac{1}{\mu_0}|B_0|^2+\frac{\partial\omega\varepsilon_g^{[0]}(\omega)}{\partial\omega}|E_0|^2+\frac{k^2}{\varepsilon_0\mu_0}\frac{\partial}{\partial\omega}\frac{\varepsilon_g^{T[2]}(\omega)}{\omega}|E_0|^2\right]=-W_{ext}-\frac{\omega}{2}\mathrm{Im}\left[\varepsilon_g^{[0]}(\omega)+\frac{k^2\varepsilon_g^{T[2]}(\omega)}{\omega^2\varepsilon_0\mu_0}\right]|E_0|^2. \quad (15)$$

Here we have incorporated to the energy flux and the energy density additional contributions that come from W_{ind}, and have written the energy theorem directly in terms of the second order coefficient $\varepsilon_g^{T[2]}(\omega)$ of the transverse response. Here

$$\vec{S}_{TRANS}=\frac{1}{2}\mathrm{Re}\left[\frac{1}{\mu_0}(\vec{E}_0\times\vec{B}_0^*)(1-\varepsilon_g^{T[2]*}(\omega)/\varepsilon_0)\right] \quad (16)$$

is the expression for the Poynting vector for transverse fields which shows explicitly that the contributions coming from the presence of matter have a non-local origin, even in the long wavelength limit. Next we will connect this coefficient with the more traditional view of the magnetic effects in terms of the magnetic permeability.

4. The $\varepsilon\mu$ scheme

We have shown that in the long wavelength limit, the energy theorem can be clearly expressed in terms of $\varepsilon_g^{[0]}$, $\varepsilon_g^{L[2]}$ and $\varepsilon_g^{T[2]}$. Here we will write it in terms of the electric permittivity ε and the magnetic permeability μ, in what we call the $\varepsilon\mu$ scheme. The problem with this scheme lies in the fact that the definitions of ε and μ are not unique and there is no general agreement in how to define them.

In the $\varepsilon\mu$ scheme one starts by splitting the induced current

$$\vec{J}^{ind}(\vec{r},t)=\frac{\partial\vec{P}(\vec{r},t)}{\partial t}+\nabla\times\vec{M}(\vec{r},t), \quad (17)$$

in terms of two material fields: the polarization $\vec{P}$ and the magnetization $\vec{M}$ fields. Note that Eq. (17) does not provide a unique definition of $\vec{P}$ and $\vec{M}$ and specific choices might depend on the choice of particular models. Since in our case $\vec{J}_{ind}$ is quasi-monochromatic (see Eq. (11)), then $\vec{P}$ and $\vec{M}$ should be also quasi-monochromatic, and we may express the response of $\vec{P}$ and $\vec{M}$ in several different ways. Here we choose, as an example,

$$\vec{P}_0(\vec{r},t)=[(\varepsilon^L(k,\omega)-\varepsilon_0)\overleftrightarrow{P}^L+(\varepsilon^T(k,\omega)-\varepsilon_0)\overleftrightarrow{P}^T]\cdot\vec{E}_0(\vec{r},t) \quad (18)$$

$$\vec{M}_0(\vec{r},t)=\left[\frac{1}{\mu_0}-\frac{1}{\mu(k,\omega)}\right]\vec{B}_0(\vec{r},t), \quad (19)$$

where $\vec{P}_0$ and $\vec{M}_0$ denote the amplitudes of $\vec{P}$ and $\vec{M}$, respectively, and we account for the possibility that the polarization field $\vec{P}$ responds non-locally and differently to the longitudinal and transverse components of the electric field. Here $\varepsilon^L(k,\omega)$ and $\varepsilon^T(k,\omega)$ denote the longitudinal and transverse dielectric responses (not to be confused with the corresponding quantites of the generalized dielectric response), while the magnetic permeability $\mu(k,\omega)$ is defined through the non-local response of $\vec{M}$ to $\vec{B}$, which is a transverse field. For consistency, we demand $\varepsilon^L(k\to0,\omega)=\varepsilon^T(k\to0,\omega)\equiv\varepsilon(0,\omega)$. In the $\varepsilon\mu$ scheme the response is characterized by three scalar functions ε^L, ε^T and μ.

If one now compares Eqs. (17)–(19) with Eqs. (1)–(3), one can identify

$$\varepsilon^L(k,\omega)=\varepsilon_g^L(k,\omega)\quad\text{and}\quad\mu(k,\omega)=\frac{1}{\frac{1}{\mu_0}-\frac{\omega^2}{k^2}[\varepsilon_g^T(k,\omega)-\varepsilon^T(k,\omega)]}. \quad (20)$$

This definition regards the induced magnetism in the system, enclosed in the magnetic permeability $\mu(k,\omega)$, as a non-local transverse effect. By making a long wavelength expansion in ε^L and ε^T in Eq. (20) one can identify

$$\varepsilon_g^{[0]}(\omega)=\varepsilon^L(0,\omega),\quad\varepsilon_g^{L[2]}(\omega)=\varepsilon^{L[2]}(\omega)\quad\text{and}\quad\varepsilon_g^{T[2]}(\omega)=\varepsilon^{T[2]}(\omega)+\varepsilon_0\left(1-\frac{\mu_0}{\mu(0,\omega)}\right) \quad (21)$$

where $\varepsilon^{L[2]}(\omega)$ and $\varepsilon^{T[2]}(\omega)$ are the second order expansion coefficients of ε^L and ε^T in powers of k^2/k_0^2, similar to the coefficients used in Eq. (4) for the longitudinal and transverse components of the *generalized* dielectric response ε_g^L and ε_g^T. Now we substitute the expression for $\varepsilon_g^{T[2]}(\omega)$ into the expression for the Poynting's vector given in Eq. (16) and write

$$\vec{S}_{TRANS}=\frac{1}{2}\mathrm{Re}\left[\vec{E}_0\times\left(\frac{\vec{B}_0^*}{\mu_0}-\vec{M}_0^*\right)-\vec{E}_0\times\frac{\vec{B}_0^*}{\mu_0}\frac{\varepsilon^{T[2]*}(\omega)}{\varepsilon_0}\right] \quad (22)$$

where

$$M_0=\left(\frac{1}{\mu_0}-\frac{1}{\mu(0,\omega)}\right)\vec{B}_0 \quad (23)$$

is identified as a local magnetization, that is, we regard a non-local electric response as a local magnetic response. Nevertheless, it is now clear that the explicit form for the expression of the Poynting vector in the $\varepsilon\mu$ scheme is not unique but depends on the choice made for the polarization and magnetization fields. This freedom arises from the ambiguity in the definition of these same fields.

One common choice is to take $\varepsilon^{T[2]*}(\omega)=0$, demanding a local transverse dielectric response to second order in k, so that all transverse non-locality is attached to $\mu(k,\omega)$. Then, following Eq. (22), the expression for the Poynting vector in the long wavelength limit, can be written as $\vec{S}_{TRANS}=(1/2)\mathrm{Re}\,\vec{E}_0\times H_0^*$, which is the usual textbook expression in terms of the $\vec{H}$ field, defined as $\vec{H}=\vec{B}/\mu_0-\vec{M}$.

Another common choice is to take $\varepsilon^{T[2]*}(\omega)=\varepsilon^{L[2]*}(\omega)$, decreeing that the non-local permittivity is scalar to second order, with the same response to the longitudinal and transverse fields. According to Eq. (22), for this choice one has to modify the usual expression for the Poynting vector $\vec{S}_{TRANS}=(1/2)\mathrm{Re}\,\vec{E}_0\times H_0^*$ by adding an extra term that "compensates" the introduction of a longitudinal response in what is strictly a transverse effect. Notice that the values of $\mu(0,\omega)$ are not the same for these two choices.

Finally, we want to add a closing comment saying that we have treated here the problem of energy transport of a quasi-monochromatic electromagnetic field in the presence of external sources. This allowed us to keep k and ω as independent variables. We did not approach the problem of energy transport by free-propagating modes, but we believe it can be dealt in a similar

Please cite this article as: R.G. Barrera, et al., Physica B (2010), doi:10.1016/j.physb.2010.01.004

manner, suppressing the external sources and taking into account that the wave vector $\vec{k}$ and $k=\sqrt{\vec{k}\cdot\vec{k}}$ become frequency-dependent complex quantities.

5. Conclusions

We used the formalism developed for the treatment of the non-local effective media associated to colloids to find a general expression for the energy theorem in the long wavelength limit, in terms of parameters associated to the *generalized* non-local response of the system. We found an explicit expression for Poynting's vector in terms of these non-local parameters and showed that there are many ways of writing it in terms of an electric permittivity ε and a magnetic permeability μ, depending on the choice taken to define them. This freedom of choice comes from the ambiguity in the definition of the material fields. The importance of having a correct definition of the Poynting vector is important for the treatment of refraction. This approach can be extended to finite wave vectors, non-negligible dissipation and turbidity, as well as problems related to negative refraction.

Acknowledgments

We acknowledge the economic support of Consejo Nacional de Ciencia y Tecnología (Mexico) through Project no. 82073 and DGAPA-UNAM through Project IN120909 (WLM).

References

[1] L.D. Landau, L.P. Pitaevskii, E.M. Lifshitz, Electrodynamics of Continuous Media, second ed., vol. 8 (Course of Theoretical Physics), Butterworth-Heinemann, January 1984 (Sec. 80).
[2] See for example: L. Tsang, J.A. Kong, K.-H. Ding, Scattering of Electromagnetic Waves Theories and Applications, Wiley, New York, 2000.
[3] R.G. Barrera, A. Reyes-Coronado, A. García-Valenzuela, Phys. Rev. B 75 (2007) 184202 [1–19].
[4] See for example: J. Pendry, D. Smith, Physics Today (2003) December; Focus on Negative Refraction, New J. Phys. 7 (2005).
[5] N. García, M. Nieto-Vesperinas, Optics Letters 27 (2002) 885.
[6] V.A. Markel, Optics Express 16 (2008) 19152.
[7] H.J. Lezec, J.A. Dionne, H.A. Atwater, Science 316 (2007) 430.
[8] M. Bredov, V. Rumiantsev, I. Toptiguin, Electrodinámica Clásica, MIR, Moscú, 1986 (Part II, Chapter I, Sec. 8).

Please cite this article as: R.G. Barrera, et al., Physica B (2010), doi:10.1016/j.physb.2010.01.004